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1989 J. Phys.: Condens. Matter 1 9387

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J. Phys.: Condens. Matter **1** (1989) 9387–9400. Printed in the UK

Dynamic conductivity for a disordered 2D electron system in a strong magnetic field

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Received 2 March 1989

Abstract. The diagonal part of the frequency-dependent conductivity tensor for the lowest Landau level of a disordered two-dimensional electron system is calculated with a variational method. This becomes rigorous, if the Fermi energy ε is deep in the tails of the density of states, i.e. in two cases: for fixed ε in the regime of weak disorder; and in the hydrodynamic limit $|\varepsilon| \to \infty$. In both cases, the non-linear variational equations are solved and the resulting frequency dependence of the conductivity is thus analytically determined. As the frequency approaches zero in the hydrodynamic limit, the conductivity vanishes.

1. Introduction

In the physics of disordered systems, the problem of localisation of electronic states under the influence of a magnetic field has attracted a lot of interest in the last years due to the discovery of the quantum Hall effect (QHE) (see e.g. Ando 1985). Concerning the transport properties of such disordered systems in two dimensions, most of the work focuses on the DC conductivity (one of the few exceptions is the application of percolation theory by Joynt 1985). However, for two reasons it would be desirable to determine the AC conductivity at low frequencies ω. Theoretically, the properties of the localised quantum states determine how the parallel conductivity approaches zero as ω is lowered (at zero temperature). Thus a knowledge of this behaviour contributes to our picture of localisation at finite magnetic field B and allows us to check the naive invocation of the $B = 0$ localisation theory in a heuristic explanation of the vanishing DC parallel conductivity in the QHE regime. Secondly, given the precision of the measurement of the conductivity tensor $\boldsymbol{\sigma}$ in the quantum Hall effect (see e.g. Yoshihiro *et al* 1985), it is important to investigate any possible cause (finite ω) for a theoretical deviation from the values

$$\sigma^{xx} = 0 \qquad \sigma^{xy} = (e^2/2\pi\hbar)\ \text{integer}$$

in the QHE regime.

To get a result for the conductivity is a severe theoretical problem, since one has to go beyond perturbation theory in order to treat the effect of localisation properly. Yet there is no rigorous analytic theory for $\boldsymbol{\sigma}(\varepsilon_f, \omega)$, not even for the simplest model of electrons moving in a strong magnetic field B and a disorder potential. But, in particular

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in view of dimension two being critical for localisation, theoretical predictions should be calculated from the given model without further approximations. At zero magnetic field $B = 0$, at least the asymptotic behaviour of σ^{xx} is rigorously known for electrons moving in a disorder potential. Houghton *et al* (1980) calculated with an instanton method the conductivity $\sigma^{xx}(\varepsilon_f, \omega)$ for Fermi energies $\varepsilon_f \rightarrow -\infty$, asymptotically deep in the region of localisation. They found the same result as Mott many years before, whose argument (see Mott and Davis 1979) uses the concept of localised states.

The purpose of this work is to devise an instanton method for the conductivity of a corresponding model including a strong magnetic field. As a first test, such a method has recently been applied successfully to the problem of the density of states (Apel 1987). The result generalises in the asymptotic regime ($\varepsilon_f \rightarrow -\infty$) the exact result of Wegner (1983) to a general gaussian distribution of the random disorder potential and, of course, coincides with it for a white noise distribution. The instanton method used here deals directly with the (function-space) integral over the potential, i.e. with the disorder average of the physical quantity in question: here the conductivity. There is another more frequently used approach which starts out from representing, say, the conductivity with an auxiliary function-space integral of replica or supersymmetric type, then averages over the disorder potential, and finally treats the resulting non-linear field theory. In comparison, the present method avoiding any auxiliary fields is more tractable and more transparent since the elementary field is the physical potential. The resulting 'instanton potential' is the particular potential which yields the main contribution to the conductivity in the disorder average.

In principle, the instanton method is applicable to the full problem including all Landau levels. The calculation involves a variation and leads finally to two coupled non-linear differential equations which are to be solved numerically. Now, restricting from the beginning the Hilbert space of the electronic states to the lowest Landau level (by taking the high-field limit) allows an analytical solution of these equations. As a result, the leading asymptotic expression for the conductivity can be given analytically as well. Thus the high-field limit greatly simplifies the calculations while the interesting non-perturbational physics of localisation is still kept in the model. That is the reason why in this work the configuration space is restricted to states of the lowest Landau level, and thus only the high-field limit $B \rightarrow \infty$ is considered. Even in this limit, there is still no exact solution of the transport properties of the model for the whole range of the Fermi energy and frequency.

The outline of the paper is as follows. In the next section the instanton method is developed and the limits are derived where it becomes rigorous. The instanton equations are solved and the expression for σ^{xx} is given, in the limit of weak disorder and in the hydrodynamic limit, in §§ 3 and 4 respectively. The results, in particular the limit of zero external frequency, are discussed in the conclusion.

2. Instanton method

In this section the expression for the conductivity is formulated for a two-dimensional system of electrons moving in a strong magnetic field and a disorder potential. The instanton equations are then derived. Kubo's formula defines the frequency (ω) and wavevector-dependent conductivity tensor in terms of a current-density current-density response function. In this work it is intended to deal only with states of a single (the lowest) Landau level. Because the current operator yields non-vanishing matrix elements only between neighbouring Landau levels, one had better use another equivalent formulation. Fortunately, the longitudinal part of the conductivity tensor can be related to the density–density response function via the equation of charge conservation (usually called the Einstein relation), and this function is well defined for a single Landau level. Thus one gets the following representation of the real part of the ω-dependent longitudinal conductivity at wavevector zero:

$$\mathrm{Re}\,\sigma^{xx}(\boldsymbol{q}=0,\omega)=\frac{e^2}{2\pi\hbar}\int \mathrm{d}\varepsilon\,\frac{f(\varepsilon-\omega/2)-f(\varepsilon+\omega/2)}{\omega}\,\sigma_{\mathrm{l}}(\varepsilon,\omega) \tag{2.1}$$

where e denotes the electronic charge, $\hbar$ is Planck's constant and $f(\varepsilon)$ is the Fermi function. $\sigma_{\mathrm{l}}(\varepsilon, \omega)$ is given by the density–density response function

$$\sigma_{\mathrm{l}}(\varepsilon,\omega)=\lim_{q\to 0}2\pi^2(\omega^2/q^2)\int \mathrm{d}^2r\,\exp[-\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}-\boldsymbol{r}')]K(\boldsymbol{r},\boldsymbol{r}';\varepsilon,\omega) \tag{2.2}$$

where K is the disorder-averaged two-particle spectral function

$$K(\boldsymbol{r},\boldsymbol{r}';\varepsilon,\omega)=\langle\!\langle \boldsymbol{r}|\delta(\varepsilon-\omega/2-H)|\boldsymbol{r}'\rangle\langle \boldsymbol{r}'|\delta(\varepsilon+\omega/2-H)|\boldsymbol{r}\rangle\!\rangle_V. \tag{2.3}$$

The single-particle Hamiltonian H contains the magnetic field and the disorder potential. For the reasons given in the introduction, only the states of the lowest Landau level are kept here. Thus H is taken to be the projection of the complete system onto the lowest Landau level

$$H=P\{\tfrac{1}{2}[(1/\mathrm{i})\nabla-\boldsymbol{A}]^2+V(\boldsymbol{r})\}P \tag{2.4}$$

P being the corresponding projector. Correspondingly in equation (2.3), the states are restricted:

$$|\boldsymbol{r}\rangle=\sum_m \psi_m(\boldsymbol{r})|m\rangle$$

where $\psi_m(\boldsymbol{r})$ is the angular momentum basis of the lowest Landau level. The symmetric gauge for the vector potential $\boldsymbol{A}=\frac{1}{2}(y,-x)$ is adopted. All energies and lengths are measured in units of the cyclotron energy and the magnetic length respectively. The disorder potential $V(\boldsymbol{r})$ is taken to be a gaussian random potential with $\langle V(\boldsymbol{r})\rangle_V=0$ and

$$\langle V(\boldsymbol{r})V(\boldsymbol{r}')\rangle_V=2\pi g\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{2.5}$$

Here only the simplest case of a white noise correlation is considered. The dimensionless coupling g measures the strength of the impurity scattering. Its relation to the scattering time τ in a Born approximation at zero magnetic field is

$$g=1/2\pi\omega_{\mathrm{B}}\tau. \tag{2.6}$$

Because of the above scaling of the units of energy and length, the cyclotron frequency ω_{B} appears in equation (2.6); τ is independent of B. With the above properties of the distribution of V, σ_{l} is a function only of $|(\varepsilon-1/2)/\sqrt{g})|$ and $|\omega/\sqrt{g}|$ (put $H=\frac{1}{2}P+PVP$ in equation (2.3), scale V with $\sqrt{g}$ and use invariance under rotations, magnetic translations and $V\to-V$). Thus $\omega>0$ and $\frac{1}{2}-\varepsilon>0$ is assumed in the following.

Next the application of the instanton method is shown. Here, as in the treatment of the density of states (Apel 1987), one considers an average of a spectral function over the disorder potential $V(\boldsymbol{r})$, in this case K (equation (2.3)). In terms of exact eigenfunctions Φ_α and eigenvalues ε_α of H one has

$$K(\boldsymbol{r},\boldsymbol{r}';\varepsilon,\omega)=\int \mathfrak{D}[V]\exp\left[-(1/4\pi g)\int \mathrm{d}^2r V(\boldsymbol{r})^2\right]\sum_{\alpha\beta}\Phi_\alpha(\boldsymbol{r})\Phi_\alpha^*(\boldsymbol{r}')$$
$$\times\,\Phi_\beta(\boldsymbol{r}')\Phi_\beta^*(\boldsymbol{r})\delta(\varepsilon-(\omega/2)-\varepsilon_\alpha)\delta(\varepsilon+(\omega/2)-\varepsilon_\beta). \tag{2.7}$$

The eigenfunctions and eigenvalues still depend on V. The main contribution to this function-space integral over $V(\boldsymbol{r})$ comes from a specific potential V^{I} which maximises the exponential weight and in addition fulfils the restrictions that $H = \frac{1}{2}P + PVP$ has eigenvalues $\varepsilon \pm \omega/2$. The optimum potential V^{I} is called the instanton potential. Approximating the integral (2.7) by the value of its integrand at $V = V^{\mathrm{I}}$ yields the exponential dependence of σ_1 on ω and ε. For the pre-exponential factor, one has to include in a second step the gaussian fluctuations around V^{I}. This 'instanton approximation' of the integral K becomes asymptotically exact when the fluctuations $V - V^{\mathrm{I}}$ are small. Equation (2.7) suggests that this happens in the limit $g \to 0$, the 'limit of weak disorder' of Houghton *et al* (1980). Since σ_1 depends only on $(\frac{1}{2} - \varepsilon)/\sqrt{g}$ and $\omega/\sqrt{g}$, the corrections to σ_1 in the limit of weak disorder are expected to be of the order of $\sqrt{g}/(\frac{1}{2} - \varepsilon)$, $\sqrt{g}/\omega$. There is yet another limit in which the fluctuations around V^{I} become small, namely $(\frac{1}{2} - \varepsilon) \to \infty$ (rescale V by $(\frac{1}{2} - \varepsilon)$). This second limit corresponds to the 'hydrodynamic limit' of Houghton *et al* (1980). Both limits are identical for the density of states, which only depends on the single variable $(\frac{1}{2} - \varepsilon)/\sqrt{g}$. In the present calculation of the conductivity, however, both limits are drastically different, as they are in the corresponding work for zero magnetic field by Houghton *et al* (1980). Only in the hydrodynamic limit, where the corrections to the leading term in σ_1 are expected to be of the order of $\sqrt{g}/\varepsilon$, ω/ε, can the low-frequency behaviour still be determined.

Having the two limits identified, in which the present instanton method becomes asymptotically exact, the equations for the optimum potential V^{I} are now derived. V^{I} is determined by a variation of the exponent of equation (2.7) with the (two) restrictions that PVP has two fixed eigenvalues, $\varepsilon - \frac{1}{2} - \omega/2$ and $\varepsilon - \frac{1}{2} + \omega/2$. Adopting two Lagrange multipliers Λ_{g} and Λ_{u}, one gets from varying in the integrand of equation (2.7)

$$(1/2\pi g)V^{\mathrm{I}}(\boldsymbol{r}) - \Lambda_{\mathrm{g}}|\Phi^{\mathrm{Ig}}(\boldsymbol{r})|^2 - \Lambda_{\mathrm{u}}|\Phi^{\mathrm{Iu}}(\boldsymbol{r})|^2 = 0. \tag{2.8}$$

The two eigenfunctions $\Phi^{\mathrm{Ig,u}}$ lie in the lowest Landau level and obey the Schrödinger equation. Expanding $\Phi^{\mathrm{Ig}}(\boldsymbol{r})$ (and correspondingly $\Phi^{\mathrm{Iu}}(\boldsymbol{r})$) in angular momentum eigenfunctions of the lowest Landau level

$$\Phi^{\mathrm{Ig}}(\boldsymbol{r}) = \sum_{m=0}^{\infty} \Phi_m^{\mathrm{Ig}} \psi_m(\boldsymbol{r}) \tag{2.9}$$

and inserting the instanton potential V^{I} from equation (2.8) in the Schrödinger equation, one finds the following two coupled non-linear equations for the coefficients of the wavefunctions:

$$(\tfrac{1}{2} - \varepsilon + \omega/2)\Phi_m^{\mathrm{Ig}} + \sum_n V_{mn}^{\mathrm{I}} \Phi_n^{\mathrm{Ig}} = 0 \tag{2.10a}$$

$$(\tfrac{1}{2} - \varepsilon - \omega/2)\Phi_m^{\mathrm{Iu}} + \sum_n V_{mn}^{\mathrm{I}} \Phi_n^{\mathrm{Iu}} = 0 \tag{2.10b}$$

where the potential is in turn determined by the wavefunctions

$$V_{mn}^{\mathrm{I}} = g/2 \sum_{m'n'} I_{mm'n'n} (\Lambda_{\mathrm{g}} \Phi_{m'}^{\mathrm{Ig}*} \Phi_{n'}^{\mathrm{Ig}} + \Lambda_{\mathrm{u}} \Phi_{m'}^{\mathrm{Iu}*} \Phi_{n'}^{\mathrm{Iu}}). \tag{2.10c}$$

The Lagrange multipliers Λ_{g} and Λ_{u} are also found from equation (2.10) by normalising the eigenfunctions Φ^{Ig} and Φ^{Iu}. The coupling of the non-linear term is

$$I_{mm'n'n} = 4\pi \int \mathrm{d}^2 r\, \psi_m^*(\boldsymbol{r}) \psi_{m'}^*(\boldsymbol{r}) \psi_{n'}(\boldsymbol{r}) \psi_n(\boldsymbol{r})$$

$$= \delta_{m+m'n'+n} \left[\frac{1}{2^{m+m'}} \binom{m+m'}{m} \frac{1}{2^{n+n'}} \binom{n+n'}{n} \right]^{1/2}. \tag{2.11}$$

The equations (2.10) solve the question of which potential V^{I} yields the maximum contribution to the averaged conductivity (2.7). They have spatially localised solutions $\Phi^{\mathrm{Ig,u}}(\boldsymbol{r})$ (see below) which are called instantons. Equations (2.10) look like mean-field equations of a φ^4 theory, similar to the corresponding equations of Houghton *et al* (1980) at $B = 0$. They differ crucially, though, from the latter and thus the solution is different. In the case of a strong magnetic field considered here, the kinetic energy is quenched, i.e. the same ($\frac{1}{2}$) for all states as opposed to $p^2/2m$. The potential energy, in angular momentum representation the matrix V^{I}_{mn}, completely determines the quantum mechanical eigenvalues. In real space it becomes non-local due to the projection P on the lowest Landau level ($PV^{\mathrm{I}}P$), in contrast to the case $B = 0$ where it is a local potential. In equation (2.10), the particles described by the wave functions Φ^{Ig}, Φ^{Iu} have one degree of freedom, the angular momentum $m = 0, 1, 2 \ldots$. This reduction to one dimension resulting from the restriction to the lowest Landau level (the high-field limit) highly simplifies the problem compared to the case $B = 0$ and allows an exact solution of the equations for V^{I}. The fluctuations around V^{I} in the function-space integral, on the other hand, are apparently two-dimensional, because the function $V(\boldsymbol{r})$ to be integrated over depends on two variables. As will be shown below, however, only a one-dimensional subset of these fluctuations contributes to the resulting expression of the conductivity.

3. Solution: regime of weak disorder

In this section the limit of weak disorder $g \to 0$ is discussed. First the solution of equation (2.10) is found. Then, the gaussian fluctuations around the solution are calculated and the result for the conductivity is given. Corrections to the leading asymptotic behaviour are here expected to be of the order of $\sqrt{g}/(\frac{1}{2} - \varepsilon)$, $\sqrt{g}/\omega$.

The key to the rigorous solution of the non-linear instanton equations is the restriction that only lowest Landau level wavefunctions are admitted together with the rotational invariance of the averaged system (see the angular momentum selection rule in equation (2.11)). Solutions are wavefunctions $\Phi^{\mathrm{Ig,u}}$ which contain each only a *single* angular momentum state $m = m_{\mathrm{g}}$, m_{u} (cf the calculations of the density of states: Apel 1987). Inserting now $\Phi^{\mathrm{Ig}}_m = \delta_{mm_{\mathrm{g}}}$, and similarly for Φ^{Iu}_m, solves the m dependence in the equations (2.10) and leaves two equations for the two Lagrange multipliers Λ_{g}, Λ_{u}. Thus a set of solutions V^{I} is determined via equation (2.8), with two arbitrary positive integers m_{g} and m_{u}. The particular solution with the maximum contribution to the conductivity is found from evaluating the exponent in equation (2.7). Because the calculation of σ_{l} involves a matrix element with the position operator, m_{g} has to be equal to $m_{\mathrm{u}} \pm 1$ and then $m_{\mathrm{g}} = 0$, $m_{\mathrm{u}} = 1$ yield the smallest exponent: the maximum contribution in equation (2.7). Finally, the solution for the instanton potential is

$$V^{\mathrm{I}}(\boldsymbol{r}) = -8\pi[\omega|\psi_0(\boldsymbol{r})|^2 + (\tfrac{1}{2} - \varepsilon - \tfrac{3}{2}\omega)|\psi_1(\boldsymbol{r})|^2]. \tag{3.1}$$

The wavefunctions $\Phi^{\mathrm{Ig,u}}(\boldsymbol{r}) = \psi_{0,1}(\boldsymbol{r})$ are localised around $\boldsymbol{r} = 0$. V^{I} is a single rotational invariant potential well centred at the origin. As r increases, $V^{\mathrm{I}}(\boldsymbol{r})$ decays as $\exp(-\frac{1}{2}r^2)$. Since the distribution of the disorder potential $V(\boldsymbol{r})$ is homogeneous, the application of a magnetic translation leads to a family of exact solutions of equations (2.10), $V^{\mathrm{I}}(\boldsymbol{r} - \boldsymbol{r}_0)$, with the same value of the exponent in equation (2.7) for arbitrary centre $\boldsymbol{r}_0$.

Calculating the fluctuations around $V^{\mathrm{I}}(\boldsymbol{r} - \boldsymbol{r}_0)$ to gaussian order is a standard procedure. Parametrising $V = V^{\mathrm{I}} + \delta V$,

$$V(\boldsymbol{r}) = V^{\mathrm{I}}(\boldsymbol{r} - \boldsymbol{r}_0) + \sum_{\nu} v_\nu f_\nu(\boldsymbol{r} - \boldsymbol{r}_0) \tag{3.2}$$

where f_ν is an orthonormal basis with $f_0 \propto \partial_x V^{\mathrm{I}}$, $f_1 \propto \partial_y V^{\mathrm{I}}$, and changing variables from the translational modes v_0, v_1 to $\boldsymbol{r}_0$, one gets the following expression for the conductivity (cf equation (2.7)):

$$\sigma_1(\varepsilon, \omega) = 2\pi^2\omega^2 \exp\{-(2/g)[(\tfrac{1}{2} - \varepsilon - \omega/2)^2 + \omega^2]\}$$
$$\times \int \mathcal{D}[\delta V]\delta(v_0)\delta(v_1)\left(\int d^2r f_0(\boldsymbol{r})\partial_x V^{\mathrm{I}}(\boldsymbol{r})\right)^2$$
$$\times \exp\left\{-(1/4\pi g)\int d^2r[\delta V(\boldsymbol{r})^2 + 2\delta V(\boldsymbol{r})V^{\mathrm{I}}(\boldsymbol{r})]\right\}$$
$$\times \tfrac{1}{2}\delta(\varepsilon - \omega/2 - \varepsilon_0[V^{\mathrm{I}} + \delta V])\,\delta(\varepsilon + \omega/2 - \varepsilon_1[V^{\mathrm{I}} + \delta V])$$
$$\times (1 + \mathrm{O}(\delta V)). \tag{3.3}$$

The first factor in the integrand of equation (3.3) is the Jacobean. The integrals over $\boldsymbol{r}$ and $\boldsymbol{r}_0$ were performed, which leads in the limit $q \to 0$ to the matrix elements

$$\left|\int d^2r\, \Phi_0(\boldsymbol{r})\boldsymbol{q}\cdot\boldsymbol{r}\,\Phi_1^*(\boldsymbol{r})\right| = q/\sqrt{2}(1 + \mathrm{O}(\delta V)) \tag{3.4}$$

since $\Phi_0(\boldsymbol{r}) = \Phi^{\mathrm{Ig}}(\boldsymbol{r})\,(1 + \mathrm{O}(\delta V))$ and correspondingly for Φ_1. That gives the factor $\frac{1}{2}$ in the integrand of equation (3.3). The last two δ functions allow only those fluctuations δV which yield $\varepsilon - \omega/2$, $\varepsilon + \omega/2$ for $\varepsilon_{0,1}[V^{\mathrm{I}} + \delta V]$, the two lowest eigenvalues of $H = \frac{1}{2}P + P(V^{\mathrm{I}} + \delta V)P$. Expanding these restrictions up to second-order perturbation theory in δV gives the following two conditions ($\alpha = 0, 1$):

$$\mathrm{O} = \varepsilon + (\alpha - \tfrac{1}{2})\omega - \varepsilon_\alpha[V^{\mathrm{I}}] - \int d^2r\, \psi_\alpha^*(\boldsymbol{r})\delta V(\boldsymbol{r})\psi_\alpha(\boldsymbol{r})$$
$$+ \sum_{m\neq\alpha}\left|\int d^2r\, \psi_m^*(\boldsymbol{r})\delta V(\boldsymbol{r})\psi_\alpha(\boldsymbol{r})\right|^2 \frac{1}{E_{m\alpha}} + \mathrm{O}(\delta V^3) \tag{3.5a}$$

where the energy denominators are given by $E_{m1} = E_{m0} - \omega$

$$E_{m0} = \tfrac{1}{2} - \varepsilon + \omega/2 + \int d^2r\, \psi_m^*(\boldsymbol{r})V^{\mathrm{I}}(\boldsymbol{r})\psi_m(\boldsymbol{r})$$
$$= \tfrac{1}{2} - \varepsilon + \omega/2 - (1/2^m)[2\omega + (\tfrac{1}{2} - \varepsilon - \tfrac{3}{2}\omega)(m+1)]. \tag{3.5b}$$

The first three terms in equation (3.5a) cancel, since the equations are fulfilled for $\delta V = 0$. Thus equations (3.5) determine the components of δV parallel to $|\psi_{0,1}|^2$ in terms of fluctuations δV in second order. Using these two constraints in the second term $\delta V V^{\mathrm{I}}$ of the exponent of equation (3.3) (cf equation (3.1)), one sees that all the fluctuations δV are of the order of $\sqrt{g}$ (all eigenvalues turn out to be positive, cf Appendix A1). The above analysis of the variables of σ_1 then shows that the corrections to a gaussian approximation in δV are of the order of g/ω^2 and $g/(\frac{1}{2} - \varepsilon)^2$. Collecting all terms, one gets as the main result of this section the following expression for the conductivity valid in the regime $(\frac{1}{2} - \varepsilon)^2 \gg g$, $\omega^2 \gg g$:

$$\sigma_1(\varepsilon, \omega) = [\omega(\tfrac{1}{2} - \varepsilon)/g]^2 \exp\{-(2/g)[(|\tfrac{1}{2} - \varepsilon| - |\omega/2|)^2 + \omega^2]\}$$
$$\times D\left(\left|\frac{(\omega/2)}{\frac{1}{2} - \varepsilon}\right|\right)\left[1 + \mathrm{O}\left(\frac{g}{\omega^2}, \frac{g}{(\frac{1}{2} - \varepsilon)^2}\right)\right]. \tag{3.6}$$

The fluctuation integral D calculated in the Appendix A1 still depends on the ratio of the external frequency to the distance of the energy from the band centre. D turns out to be a single product (equation (A1.3)), because only a one-dimensional subset of the fluctuations δV contributes a factor different from unity to D. As already indicated above, at this point the problem displays its one-dimensionality in the limit considered here.

4. Solution: hydrodynamic limit

Here, the hydrodynamic limit $\varepsilon \to -\infty$ of $\sigma_1(\varepsilon, \omega)$ is discussed. As in the last section, first the solution of the instanton equations (2.10) is given in this limit and then the gaussian fluctuations around the instanton potential are calculated. The corrections to the leading asymptotic behaviour are expected to be of the order of $\sqrt{g}/\varepsilon$, ω/ε.

In the hydrodynamic limit the equations (2.10) become degenerate, since $\Omega = (\omega/2)/(\frac{1}{2} - \varepsilon)$ goes to zero. Trying to extrapolate the solution of the last section to $\Omega = 0$, one finds from the calculation in Appendix A1 that one of the fluctuation eigenvalues ($m = 2$) vanishes and the corresponding integral $D_2(\Omega)$ diverges as Ω^{-2}. The corresponding eigenvector is easily identified. This mode (of the order of $\sqrt{g}/\Omega$) could be considered small compared to V^{I} (of the order of $\frac{1}{2} - \varepsilon$) only in the regime of weak disorder in the last section. But here, in the hydrodynamic limit, $\Omega \to 0$ and it has to be kept to all orders. A careful examination of the equations (2.10) now shows that the point V^{I} where the integrand in equation (2.7) becomes maximal degenerates to a whole surface as $\Omega \to 0$. That is the reason for the mode with the vanishing eigenvalue mentioned above. The degenerate solutions on this surface are made up from two functions, namely the spherical symmetric eigenfunction Φ^{Ig}_m of the last section, shifted with magnetic translations by $\boldsymbol{\xi}$ and $-\boldsymbol{\xi}$. Surprisingly, their symmetric and antisymmetric superpositions solve equations (2.10), in spite of their non-linearity. Indeed one verifies that

$$\Phi^{\mathrm{Ig}}_m(\xi) = \begin{cases} [m!\,2^m \cosh \frac{1}{2}|\xi|^2]^{-1/2}\xi^{*m} & m \text{ even} \\ 0 & m \text{ odd} \end{cases} \tag{4.1a}$$

$$\Phi^{\mathrm{Iu}}_m(\xi) = \begin{cases} 0 & m \text{ even} \\ [m!\,2^m \sinh \frac{1}{2}|\xi|^2]^{-1/2}\xi^{*m} & m \text{ odd} \end{cases} \tag{4.1b}$$

solves equations (2.10) at $\Omega = 0$ with $\Lambda_{\mathrm{g}} = 0$, $\Lambda_{\mathrm{u}} = -(\frac{1}{2} - \varepsilon)4/g$ rigorously for all values of the complex variable ξ which parametrises the surface of maxima. In real space,

$$\Phi^{\mathrm{Ig,u}}(\boldsymbol{r}, \xi) = \frac{f(\xi^* z/2)}{[2\pi f(\frac{1}{2}|\xi|^2)]^{1/2}} \exp(-r^2/4) \tag{4.1c}$$

with $f = \cosh$ and $f = \sinh$ respectively (and $z = x + \mathrm{i}y$). The corresponding potential

$$V^{\mathrm{I}}(\boldsymbol{r}, \xi) = -(\tfrac{1}{2} - \varepsilon)8\pi \frac{|\sinh(\xi^* z/2)|^2}{2\pi \sinh \frac{1}{2}|\xi|^2} \exp(-r^2/2) \tag{4.2}$$

yields an exponent of $(2/g)(\frac{1}{2} - \varepsilon)^2$ in equation (2.7) independent of ξ. Figure 1 shows $V^{\mathrm{I}}(r, \xi)$ for fixed $\xi = 2.5\mathrm{i}$. It is an inversion symmetric double-well potential. For large $|\xi|$, twice the modulus of ξ gives the distance between the wells and its phase gives the direction from the origin to one of the minima. As ξ decreases, the two minima flatten and merge and the potential well goes over in the rotational symmetric solution of the last section at $\Omega = 0$ (equation (3.1)). Linear expansion in ξ gives again the above mentioned eigenvector of the zero mode. Comparing with the work at $B = 0$, the

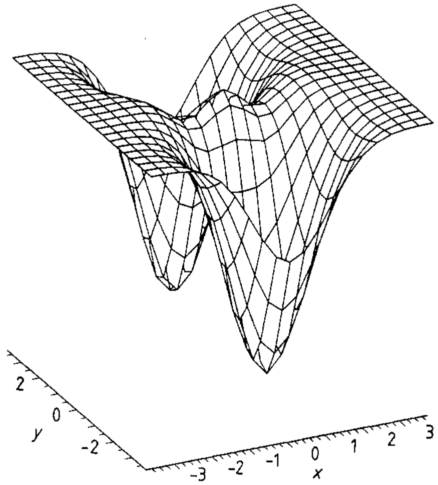

Figure 1. Plot of the instanton potential $V^{\mathrm{I}}(\boldsymbol{r}, \xi)$ (equation (4.2)) in arbitrary units over the x, y plane for $\xi = 2.5\mathrm{i}$.

corresponding potential there looks similar and is approximately given by a superposition of two $B = 0$ single-well instanton potentials separated by a distance $\boldsymbol{a}$. It should be stressed, however, that in spite of this apparent similarity the quantum mechanics in both cases is quite different, as already explained at the end of § 2. The case of strong magnetic field considered here shows the simplifying feature that an analytical expression for the double well instanton potential can be rigorously derived, thanks to the restriction to the lowest Landau level. If we now increase Ω from zero to a finite value, the degeneracy described by the ξ coordinate is lifted as will be seen below, and only double well potentials with smaller and smaller distance can still contribute to the conductivity. But because the limit $\Omega \to 0$ is to be studied in this section, it is necessary to keep the full non-linearity of the ξ coordinate. Therefore the result of the last section cannot simply be taken over to the hydrodynamic limit.

Now the gaussian fluctuations around $V^{\mathrm{I}}(\boldsymbol{r}, \xi)$ are to be calculated for fixed finite ξ. In the parametrisation $V = V^{\mathrm{I}} + \delta V$ four collective coordinates have to be singled out: an arbitrary translation ($\boldsymbol{r}_0$) and ξ;

$$V(\boldsymbol{r}) = V^{\mathrm{I}}(\boldsymbol{r} - \boldsymbol{r}_0, \xi) + \sum_{\nu} v_\nu f_\nu(\boldsymbol{r} - \boldsymbol{r}_0, \xi). \tag{4.3}$$

Here the orthonormal basis f_ν contains the translational modes f_0 and f_1, and the modes related to the collective coordinate ξ,

$$f_2 \propto \partial_{|\xi|} V^{\mathrm{I}}(\boldsymbol{r}, \xi) \qquad f_3 \propto \partial_{\psi} V^{\mathrm{I}}(\boldsymbol{r}, \xi)$$

where $\xi = |\xi| \exp(\mathrm{i}\psi)$. After eliminating $v_0 \ldots v_3$ in favour of $\boldsymbol{r}_0$ and ξ, the expression for the conductivity becomes (cf equation (2.7))

$$\begin{aligned} \sigma_1(\varepsilon, \omega) = 2\pi^2\omega^2 \int_0^\infty \mathrm{d}\,|\xi| \int \mathfrak{D}[\delta V] \prod_{\nu=0}^{3} \delta(v_\nu) \\ \times J(\tfrac{1}{2}|\xi|^2, \delta V) \quad \exp[-E(\tfrac{1}{2}|\xi|^2, \delta V)] \quad C(\tfrac{1}{2}|\xi|^2, \delta V) \\ \times \delta(\varepsilon - \omega/2 - \varepsilon_0[V^{\mathrm{I}} + \delta V]) \quad \delta(\varepsilon + \omega/2 - \varepsilon_1[V^{\mathrm{I}} + \delta V]). \end{aligned} \tag{4.4}$$

Here the Jacobean J, the exponent E and the eigenvalues ε_0, ε_1 do not depend on the position $\boldsymbol{r}_0$ and orientation ψ of the instanton solution V^{I}. Thus integrating over $\boldsymbol{r}_0$ and

ψ leads to the following expression for C, the square of the matrix element of the position operator:

$$C(\tfrac{1}{2}|\xi|^2, \delta V) = \int_0^\pi \mathrm{d}\psi \left| \int \mathrm{d}^2 r \Phi_0^*(\boldsymbol{r}, \xi, \delta V) \hat{\boldsymbol{e}} \cdot \boldsymbol{r} \, \Phi_1(\boldsymbol{r}, \xi, \delta V) \right|^2 \tag{4.5}$$

where $\hat{e}^2 = 1$. Next the expressions for J, C and E are given in leading order in the expansion with respect to δV. Since V^{I} is proportional to $(\frac{1}{2} - \varepsilon)$, that is an expansion in $(\frac{1}{2} - \varepsilon)^{-1}$. For J, the leading order is already given at $\delta V = 0$:

$$J(x, 0) = 16\pi^2(\tfrac{1}{2} - \varepsilon)^4 \sqrt{2x}\,(\cosh x \sinh x - x)/\sinh^2 x. \tag{4.6}$$

For C, one needs the wavefunctions of the next, first order degenerate $(\varepsilon_{0,1}[V^{\mathrm{I}}] = \varepsilon)$, perturbation theory. In this order the fluctuations

$$\begin{aligned} &f_4 \propto |\Phi^{\mathrm{Iu}}|^2 && f_5 \propto |\Phi^{\mathrm{Ig}}|^2 - |\Phi^{\mathrm{Iu}}|^2 \\ &f_6 \propto \mathrm{Re}\, \Phi^{\mathrm{Ig}*}\Phi^{\mathrm{Iu}} && f_7 \propto \mathrm{Im}\, \Phi^{\mathrm{Ig}*}\Phi^{\mathrm{Iu}} \end{aligned} \tag{4.7}$$

contribute. In equation (4.7), the arguments $(\boldsymbol{r}, \xi)$ are suppressed for brevity. The functions $f_0 \ldots f_7$ are mutually orthogonal. The energy eigenvalues are to first order

$$\begin{aligned} \varepsilon_{0,1} &= \varepsilon + v_4' + \frac{v_5'}{2\cosh} \\ &\mp \left[\left(\frac{v_5'}{2\cosh}\right)^2 + \frac{1+\tanh}{2} v_6'^2 + \frac{1-\tanh}{2} v_7'^2\right]^{1/2} + \mathrm{O}\left(\frac{1}{\frac{1}{2} - \varepsilon}\right) \end{aligned} \tag{4.8}$$

with $v_\nu' = v_\nu/(8\pi)^{1/2}$. The hyperbolic functions have the argument $\frac{1}{2}|\xi|^2$. They arise from normalisations and inner products of different products of $\Phi^{\mathrm{Ig,u}}(\boldsymbol{r}, \xi)$. The two last δ functions in equation (4.4) constrain the fluctuations δV to those which yield the eigenvalues $\varepsilon \mp \omega/2$. Thus v_5, v_6 and v_7 are constrained to be on an ellipsoid and v_4 is given by v_5. Introducing spherical coordinates

$$\begin{aligned} v_5' &= 2\cosh x \qquad v\cos\theta \\ v_6' &= \left(\frac{2}{1 + \tanh x}\right)^{1/2} v \sin\theta \cos\varphi \\ v_7' &= \left(\frac{2}{1 - \tanh x}\right)^{1/2} v \sin\theta \sin\varphi \end{aligned} \tag{4.9}$$

and expanding the wavefunctions $\Phi_{0,1}$ (in first-order perturbation theory) in $\Phi^{\mathrm{Ig,u}}$ (equation (4.1*c*)), one gets from equation (4.5) after some algebra

$$\begin{aligned} C(x, \delta V) &= (\pi/2)x[(1 + \cos^2\theta)\coth 2x - (1 - \cos^2\theta)\cos 2\varphi] \\ &\quad \times \left[1 + \mathrm{O}\left(\frac{1}{\frac{1}{2} - \varepsilon}\right)\right]. \end{aligned} \tag{4.10}$$

For $\xi \to \infty$, C increases as $x \propto |\xi|^2$, consistent with the meaning of $2|\xi|$ as the distance between the two minima of V^{I}. For E, one needs to go one order beyond equation (4.8) in the perturbational calculation of the eigenvalues, since the exponent

$$E(\tfrac{1}{2}|\xi|^2, \delta V) = (2/g)(\tfrac{1}{2} - \varepsilon - v_4')^2 + (4\pi g)^{-1} \sum_{\nu \neq 4} v_\nu^2 \tag{4.11}$$

(use $V^{\mathrm{I}} \propto f_4$) contains the term $(\frac{1}{2} - \varepsilon)\, v_4$ and thus the order $(\frac{1}{2} - \varepsilon)^{-1}$ in the constraint for v_4 (and v_5) has to be determined. The resulting correction to E is quadratic in the fluctuations δV (just as in the limit of small disorder in the last section, cf equation (3.5)). Thus, all fluctuations δV except v_4, v_5, v_6 and v_7 are quadratic. If we integrate the latter modes separately with the parametrisation (4.9), $\cos\theta = u$, the last two δ functions in equation (4.4) determine v_4 and v $(=\omega/2)$. Collecting all factors, one gets finally as the main result of this section the following expression for the conductivity valid in the regime $(\frac{1}{2}-\varepsilon)^2 \gg g$, $(\frac{1}{2}-\varepsilon)^2 \gg \omega^2$:

$$\begin{aligned}\sigma_1(\varepsilon,\omega) = {} & [\omega(\tfrac{1}{2}-\varepsilon)/g]^4 \int_0^\infty \mathrm{d}x \int_{-1}^{1} \mathrm{d}u \int_{-\pi}^{\pi} \frac{\mathrm{d}\varphi}{2\pi} \\ & \times \coth^2 x\,(\cosh x \sinh x - x) \\ & \times x[(1+u^2)\coth 2x - (1-u^2)\cos 2\varphi] \\ & \times \exp\{(-2/g)[(\tfrac{1}{2}-\varepsilon+\tfrac{1}{2}\omega u)^2 + \omega^2 \tilde{E}(x,u,\varphi)]\} \\ & \times D_h(x)\,[1 + \mathrm{O}(|\omega/\varepsilon|, g/\varepsilon^2)]\end{aligned} \tag{4.12}$$

with

$$\tilde{E}(x,u,\varphi) = u^2\cosh^2 x + \frac{1-u^2}{2}\left[\frac{E_x(x)\cos^2\varphi}{1+\tanh x} + \frac{E_y(x)\sin^2\varphi}{1-\tanh x}\right]. \tag{4.13}$$

Here, $\tilde{E}(x,u,\varphi)$ in the exponent and the factor $D_h(x)$ result from the fluctuation integral which is evaluated in Appendix A2.

5. Conclusions

The longitudinal conductivity $\sigma_1(\varepsilon,\omega)$ was considered for a system of electrons moving in a disorder potential and a magnetic field which was assumed to be strong enough so that only states of the lowest Landau level are populated. Exact asymptotic expressions for σ_1 in the limit $(\frac{1}{2}-\varepsilon)^2 \gg g$ were derived in two different regimes (the electronic energy ε (and also frequency ω) are measured in units of the cyclotron energy (frequency); g is proportional to the inverse scattering time (equation (2.6)). For $\omega^2 \gg g$, the limit of weak disorder, the result for σ_1 is given by equation (3.6). In the asymptotic region studied here, it becomes—as the density of states—exponentially small. The low-frequency behaviour of the conductivity can be studied in the other regime, the hydrodynamic limit, $(\frac{1}{2}-\varepsilon)^2 \gg \omega^2$, where the result for σ_1 is given by equation (4.12). Now, the integrals in equation (4.12) are evaluated to get the leading term for $g \gg \omega^2$. The second term of the exponent in equation (4.12) renders the x integral convergent by cutting it off at $x_\omega \sim \ln g/\omega^2$ ($\tilde{E}(x,u,\varphi)$ diverges for $x \to \infty$). Thus the leading term comes from the region $x \gg 1$. Double-well potentials V^1 with a separation between the wells smaller than $(8x_\omega)^{1/2}$ contribute to equation (4.12). In evaluating the u integral one has to distinguish between two different cases because of the first term in the exponent of equation (4.12). Consider first $(1/g)|\frac{1}{2}-\varepsilon|\,|\omega| \gg 1$. Then the main contribution to the u integral in equation (4.12) comes from the region $u \simeq -1$. Thus one gets the following result for the conductivity in the regime $g \gg \omega^2 \gg g^2/(\frac{1}{2}-\varepsilon)^2$:

$$\sigma_1(\varepsilon,\omega) \propto 2(|\omega|\,|\tfrac{1}{2}-\varepsilon|^3/g^2)\ln(g/\omega^2)\exp[(-2/g)(|\tfrac{1}{2}-\varepsilon| - |\omega/2|)^2]. \tag{5.1}$$

In the second case, on the other hand, i.e. where $g \gg g^2/(\frac{1}{2}-\varepsilon)^2 \gg \omega^2$, the whole range of the u integral contributes and one has a cross-over from equation (5.1) to

$$\sigma_1(\varepsilon,\omega) \propto 8[\omega^2(\tfrac{1}{2}-\varepsilon)^4/g^3]\ln(g/\omega^2)\exp[(-2/g)(\tfrac{1}{2}-\varepsilon)^2]. \tag{5.2}$$

As in the limit of weak disorder, the conductivity becomes exponentially small in the asymptotic region. The low-frequency behaviour of σ_l leading to a vanishing DC conductivity is interpreted as the result of the localisation of the quantum electronic states. It is instructive to compare the frequency dependence in equations (5.1) and (5.2) with the corresponding result for $B = 0$:

$$\sigma_l \propto \omega^2 [\ln 1/\omega\tau]^{d+1} \qquad d = 2.$$

In the strong-field case, the behaviour of σ_l is dominated by a linear frequency dependence, equation (5.1); only for an interval of ω which shrinks to zero in the limit $(\frac{1}{2} - \varepsilon) \to \infty$ studied here does σ_l cross over to the quadratic law. The power of the logarithmic correction is 1 instead of 3 as for $B = 0$, due to the different decay of the localised states as $r \to \infty$. The scale on which ω varies is $(\omega_B/\tau)^{1/2}$ instead of $1/\tau$ (in ordinary units). The comparison shows that the conductivity behaves quite differently for a disordered system at $B = 0$ and in the strong-field limit respectively. In a recent preprint, Efetov and Marikhin (1989) study the case of a large Landau level index n $(n \to \infty)$, opposite to the limit $(n = 0)$ in the present work. They find for the conductivity the same frequency dependence with the same power of the logarithmic correction as at $B = 0$.

Acknowledgment

I am indebted to F Wegner for a discussion which inspired this work.

Appendix A1

Here the fluctuation integral D in the regime of weak disorder is calculated. Choosing the functions f_2, f_3 in the expansion (3.2) orthonormal in $\{|\psi_0|^2, |\psi_1|^2\}$ one finds, comparing equation (3.6) with equation (3.3),

$$D(\Omega) = (8\pi/16)(\tfrac{1}{2} - \varepsilon)^{-2} \left(\int d^2 r f_0 \partial_x V^{\mathrm{I}} \right)^2 \times \int \mathcal{D}[\delta V] \exp(-(\delta V M \delta V)) \delta(v_0)\delta(v_1)\delta(v_2)\delta(v_3) \qquad \text{(A1.1)}$$

with $\Omega = (\omega/2)/(\frac{1}{2} - \varepsilon)$. The quadratic form in the exponent results from using the constraint (3.5) in the term $\delta V V^{\mathrm{I}}$ in the exponent of equation (3.3) and scaling δV by $\surd(4\pi g)$:

$$(\delta V M \delta V) = \int d^2 r\, \delta V(r)^2 - 16\pi \times \left[\sum_{m \neq 0} \frac{\omega}{E_{m0}} \left| \int d^2 r\, \psi_0^*(r) \delta V(r) \psi_m(r) \right|^2 + \sum_{m \neq 1} \frac{(\frac{1}{2} - \varepsilon - \frac{3}{2}\omega)}{E_{m1}} \left| \int d^2 r\, \psi_1^*(r) \delta V(r) \psi_m(r) \right|^2 \right]. \qquad \text{(A1.2)}$$

The angular momentum eigenfunctions of the lowest Landau level $\psi_m(r)$ have the angular dependence $(x + \mathrm{i}y)^m$. Because any fluctuations δV orthogonal to the functions $\psi_0^* \psi_m$ and $\psi_1^* \psi_{m+1}$ $(m = 0, 1, 2 \ldots)$ only enter the first term in $(\delta V M \delta V)$ corresponding to $M = 1$, they cancel in the integral (A1.1) against the normalisation of the disorder average. One is then left with only the above one-dimensional subset of fluctuations. These are easily evaluated, because any linear combinations of the two functions above are orthogonal for different m. The δ functions in the integrand in equation (A1.1) only restrict the fluctuations for $m = 1$ and $m = 0$, because $f_{0,1}$ belong to $m = 1$, and $f_{2,3}$ are rotational invariant, $m = 0$. Equation (A1.2) displays nicely how the restriction of the configuration space to the lowest Landau level leads to the result that only a one-dimensional subset of fluctuations δV contributes to the conductivity in the limit considered here. All fluctuations are stable and D is given by the product of the contributions from different $m = 0, 1, 2, 3 \ldots$:

$$D(\Omega) = ((1+\Omega)(1+9\Omega)/6\Omega) \prod_{m=3}^{\infty} D_m(\Omega) \tag{A1.3}$$

with

$$D_m(\Omega) = [1 - x_m(\Omega) - y_m(\Omega) + \frac{1}{m+2} x_m(\Omega) y_m(\Omega)]^{-1}$$

and

$x_m(\Omega) = 2\omega/2^m E_{m0} \qquad y_m(\Omega) = (\frac{1}{2} - \varepsilon - \frac{3}{2}\omega)(m+2)/2^{m+1} E_{m+11}$.

The product in equation (A1.3) is a monotonic function of Ω which decreases from its maximum ($\simeq 3.16$) at $\Omega = 0$ with increasing Ω.

Appendix A2

Here, the fluctuation integral in the hydrodynamic limit is examined. To this end, one needs the eigenvalues and eigenfunctions of linear fluctuations around $V^{\mathrm{I}}(\boldsymbol{r}, \xi)$ (equation (4.2)). Since $V^{\mathrm{I}} = -(\frac{1}{2} - \varepsilon) 8\pi |\Phi^{\mathrm{Iu}}|^2$, consider the eigenvalue equation

$$\int \mathrm{d}^2 r' P(\boldsymbol{r}, \boldsymbol{r}') 8\pi |\Phi^{\mathrm{Iu}}(\boldsymbol{r}', \xi)|^2 x_k(\boldsymbol{r}', \xi) = m_k(\tfrac{1}{2}|\xi|^2) x_k(\boldsymbol{r}, \xi). \tag{A2.1}$$

Equation (A2.1) is an integral equation, non-local due to $P(\boldsymbol{r}, \boldsymbol{r}')$, the projector in the lowest Landau level. Two eigenfunctions and eigenvalues are already known, namely $x_0 = \Phi^{\mathrm{Ig}}$ and $x_1 = \Phi^{\mathrm{Iu}}$, which solve the instanton equations (2.10); $m_0 = m_1 = 1$. Due to rotation invariance, the m_k depend only on the modulus of ξ. For $\xi = 0$, when V^{I} becomes rotational symmetric, the eigenvalues can be classified according to the angular momentum k, $m_k(0) = (k+1)2^{-k}$ $(k = 0, 1, 2 \ldots)$. For $\xi \to \infty$, each eigenvalue becomes doubly degenerate, $m_{2j}(\infty) = m_{2j+1}(\infty) = 2^{-j}, j = 0, 1 \ldots$. These are the energy levels in two identical wells infinitely far separated as $\xi \to \infty$. In general, $m_k(x) \geqslant 0$. Also, $1 \geqslant m_k(x)$, as the numerical diagonalisation of equation (A2.1) in the angular momentum representation shows. In the following, it becomes necessary to use the functions

$$y_k(\boldsymbol{r}, \xi) = (8\pi/m_k)^{1/2} \Phi^{\mathrm{Iu}*}(\boldsymbol{r}, \xi)\, x_k(\boldsymbol{r}, \xi). \tag{A2.2}$$

Multiplication of equation (A2.1) with x_k^* and integration shows that y_k are orthonormal; normalisation of the x_k is assumed.

Now the expression for the fluctuation integral D is derived. Following the steps leading to equation (4.12) one needs to calculate (δV is scaled by $(4\pi g)^{1/2}$):

$$D(x, \Delta_5^2, \Delta_6^2, \Delta_7^2) = 4\pi^4 \int \mathcal{D}[\delta V] \prod_{\nu=0}^{7} \delta(v_\nu - \Delta_\nu) \exp[-(\delta V M \delta V)] \tag{A2.3}$$

with $\Delta_0 = \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0$ and

$$(\delta V M \delta V) = \int \mathrm{d}^2 r\, \delta V(\boldsymbol{r})^2 - \sum_{k=2}^{\infty} \frac{28\pi}{1 - m_k} \left| \int \mathrm{d}^2 r \Phi^{\mathrm{Iu}*} \delta V x_k \right|^2. \tag{A2.4}$$

The normalisation in equation (A2.3) is such that

$$\int \mathfrak{D}[\delta V] \exp(-(\delta V\, \delta V)) = 1.$$

Here, the inner product in the exponent is defined as usual and corresponds to the first term on the RHS of equation (A2.4). The second term in equation (A2.4) comes from the second-order perturbation theory in the constraint of v_4. In inner product notation, equation (A2.4) becomes

$$(\delta V\, M\, \delta V) = (\delta V\, \delta V) - \sum_{k=2}^{\infty} \frac{2m_k}{1 - m_k} (\delta V\, y_k)(y_k\, \delta V). \tag{A2.5}$$

Similar to before, all fluctuations δV orthogonal to the y_k and to f_ν with $\nu = 0, 1 \ldots 6, 7$ yield a factor of 1 in D. Even and odd fluctuations in $\boldsymbol{r}$ do not mix in equation (A2.5) since the parity is conserved in equation (A2.1). Thus $D = D_g D_u$. Consider first the fluctuations in the space of even functions, $\{f_2, f_3, f_4, f_5, y_{2k+1}$ for $k = 1, 2 \ldots\}$. It is easy to show that $(f_4 y_{2k+1}) = (f_5\, y_{2k+1}) = 0$ for $k = 1, 2 \ldots$. Moreover, one finds $y_3 = 1/\sqrt{2}\,(f_2 + \mathrm{i} f_3)$ with $m_3(\frac{1}{2}|\xi|^2) = \frac{1}{2}$ and thus the $f_{2,3}$ become zero modes in equation (A2.5). Finally, since δV is real and y_{2k+1} is complex, one needs mutual orthonormality of the real and of the imaginary parts of y_{2k+1} for different k to separate the modes. That can be proved and one gets

$$D_g(x, \Delta_5^2) = \exp(-\Delta_5^2) 4 \prod_{k=2}^{\infty} [1 - m_{2k+1}(x)]/[1 - 2m_{2k+1}(x)]. \tag{A2.6}$$

Next, the fluctuation integral D_u is considered, which comes from the odd parity fluctuations $\{f_0, f_1, f_6, f_7, y_{2k}$ for $k = 1, 2 \ldots\}$. Unfortunately, here the real and imaginary parts of y_{2k} are not mutually orthogonal. Writing $y_{2k} = (1/\sqrt{2})(R_k + \mathrm{i}\, I_k)$ for $k = 1, 2, \ldots$, one finds

$$\begin{aligned} (R_{k'} R_k) &= \delta_{k'k} - a_{k'}(x) a_k(x) \\ (I_{k'} I_k) &= \delta_{k'k} + a_{k'}(x) a_k(x) \end{aligned} \tag{A2.7}$$

where $a_k(x) = (m_{2k}(x) \sinh x)^{-1/2} (\psi_0 x_{2k})$ describes the non-orthogonality; x denotes $\frac{1}{2}|\xi|^2$. Fluctuations parallel to R_k still do not mix with those parallel to $I_{k'}$ ($(I_{k'} R_k) = 0$); thus $D_u = D_{uR}(x, \Delta_6^2) D_{uI}(x, \Delta_7^2)$. The calculation of $D_{uR,I}$ is straightforward but complicated by the non-orthogonality (A2.7). From equation (A2.3) it is obvious that

$$D_{uR,I}(x, \Delta^2) = \exp(-\Delta^2 E_{R,I}(x))\, \tilde{D}_{uR,I}(x). \tag{A2.8}$$

The rather lengthy analytical results for the E and $\tilde{D}_u$ in terms of norms and the product of $a_k(x)$ and another similarly defined vector $b_k(x)$ are omitted here for the sake of brevity. For an evaluation of these as a function of $x = \frac{1}{2}|\xi|^2$, the eigenvalues $m_k(x)$ and also the eigenfunctions x_{2k} of equation (A2.1) in the angular momentum representation were calculated numerically.

In summary the following result was found:

$$D(x, \Delta_5^2, \Delta_6^2, \Delta_7^2) = \exp[-\Delta_5^2 - \Delta_6^2\, E_R(x) - \Delta_7^2 E_I(x)] 4 \prod_{k=4}^{\infty} \frac{1 - m_k(x)}{1 - 2m_k(x)} \tilde{D}(x). \tag{A2.9}$$

The numerical investigation shows that the $E_{R,I}(x)$ vary monotonically between $E_{R,I}(0) = -1$ and $E_{R,I}(\infty) = +1$. The product varies between 3.16 at $x = 0$ and 4 at $x = \infty$. $\tilde{D}(x)$ is of the order of 1, $\tilde{D}(0) = \tilde{D}(\infty) = 1$.

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